Boundary conditions and amplitude of Struck String

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1 Struck string

We consider a string of length l rigidly clamped at the two end points. The string is assumed to be along $x-$ axis. The string is now excited to vibration by striking it with a small hammer of width Δx over a region bounded by $x = h$ and $x = h + \Delta x$. The trans verse waves are generated from the point of excitation, travel along opposite directions from the point of striking, get reflected from the end points and superpose with each other to produce transverse stationary waves.

Thus in this case of a struck string, the initial displacement of each point on the string is zero while the initial velocity is zero for all points except the region struck, where the velocity imparted by the hammer is assumed to be u. The general displacement at any time t at any point x is given by

$$y = \sum_{s=1}^{\infty} (A_s \cos \frac{s\pi vt}{l} + B_s \sin \frac{s\pi vt}{l}) \sin \frac{s\pi x}{l} \tag{1}$$

where A_s's and B_s's are unknown Fourier constants.

$$(y)_0 = 0 = \sum_{s=1}^{\infty} A_s \sin \frac{s\pi x}{l}$$

$\Rightarrow$ A_s's are zero.

Therefore

$$y = \sum_{s=1}^{\infty} (B_s \sin \frac{s\pi vt}{l}) \sin \frac{s\pi x}{l} \tag{2}$$

$$\therefore \dot{y} = \sum_{s=1}^{\infty} B_s \frac{s\pi v}{l} \cos \frac{s\pi vt}{l} \sin \frac{s\pi x}{l}$$

$$\therefore \dot{y_0} = \sum_{s=1}^{\infty} B_s \frac{s\pi v}{l} \sin \frac{s\pi x}{l} \tag{3}$$

In order to evaluate B_s, we multiply both sides of equation 3 by $\sin \frac{m\pi x}{l}$ (where m is an integer) and integrate the result with respect to x over the length of the string.

$$\begin{aligned} \therefore \int_0^l (\dot{y})_0 \sin \frac{m\pi x}{l} dx &= \int_0^l \sum_{s=1}^{\infty} (B_s \frac{s\pi v}{l} \sin \frac{s\pi x}{l}) \sin \frac{m\pi x}{l} dx \\ &= 0, \ when\ s \neq m \\ &= B_m . \frac{s\pi v}{l} \frac{l}{2} \ when\ s = m \\ &= B_m . \frac{m\pi v}{2} \end{aligned}$$

$$\therefore B_m = \frac{2}{m\pi v}\int_0^l \dot{y}_0 \sin\frac{m\pi x}{l}dx$$

or

$$\begin{aligned}
B_s &= \frac{2}{\pi s v}\int_0^l \dot{y}_0 \sin\frac{s\pi x}{l}dx \\
&= \frac{2}{\pi s v}\left[\int_0^h 0.\sin\frac{s\pi x}{l}dx + \int_h^{h+\Delta x} u\sin\frac{s\pi x}{l}dx + \int_{h+\Delta x}^{h} 0.\sin\frac{s\pi x}{l}dx\right] \\
&= \frac{2u}{\pi s v}\left[\frac{l}{s\pi}\cos\frac{s\pi x}{l}\right]_{h+\Delta x}^{h} \\
&= \frac{2ul}{\pi^2 s^2 v}\left[\cos\frac{s\pi h}{l} - \cos\frac{s\pi}{l}(h+\Delta x)\right] \\
&= \frac{2ul}{\pi^2 s^2 v}\left[\cos\frac{s\pi h}{l} - \cos\frac{s\pi h}{l}\cos\frac{s\pi\Delta x}{l} + \sin\frac{s\pi h}{l}\sin\frac{s\pi\Delta x}{l}\right]
\end{aligned}$$

Since Δx is small, $\cos\frac{s\pi\Delta x}{l} \approx 1$ and $\sin\frac{s\pi\Delta x}{l} \approx \frac{s\pi\Delta x}{l}$.

$$\begin{aligned}
\therefore B_s &= \frac{2ul}{\pi^2 s^2 v}\frac{s\pi\Delta x}{l}\sin\frac{s\pi h}{l} \\
&= \frac{2u\Delta x}{\pi s v}\sin\frac{s\pi h}{l}
\end{aligned}$$

Hence the general displacement of the string may be written as

$$y = \frac{2u\Delta x}{\pi v}\sum_{s=1}^{\infty}\frac{1}{s}\sin\frac{s\pi h}{l}\sin\frac{s\pi vt}{l}\sin\frac{s\pi x}{l} \tag{4}$$

As evident from equation (4), the resultant vibration of the string is represented by an infinite series which is a slowly convergent one.

Let $u\Delta x = A =$ constant.

$$y = \frac{2A}{\pi v}\sum_{s=1}^{\infty}\frac{1}{s}\sin\frac{s\pi h}{l}\sin\frac{s\pi vt}{l}\sin\frac{s\pi x}{l} \tag{5}$$

$$y \propto \frac{1}{s}, \tag{6}$$

The effective contribution to y comes from large number of harmonics. But in the case of plucked string $y \propto \frac{1}{s^2}$. So a relatively smaller amount of harmonics will contribute to the resultant vibration. We know that a sound will be more melodious if it contains a large number of harmonics. For this reason the sound emitted from the struck string instrument will be more melodious than that emitted from a plucked string one.

1.1 Young-Helmholtz's Law

If

$$\sin\frac{s\pi h}{l} = 0,\ y = 0 \tag{7}$$

$\Rightarrow$ the sth mode of vibration will be absent in the resultant vibration. Now

$$\sin \frac{s\pi h}{l} = 0 \Rightarrow \frac{s\pi h}{l} = n\pi \tag{8}$$

where $n = 1, 2, 3.....$ or

$$s = n\frac{l}{h}$$

If

$$\frac{l}{h} = p, s = np$$

Hence, if the string be struck at point $x = h$ such that $\frac{l}{h} = p$, the p th, $2p$th, $3p$th etc mode of vibration will not be present. Thus in this case, Young-Helmholtz's law states that those harmonics which have a node at the struck point will be absent from the resultant vibration of the string.

1.2 Stringed musical instruments

The stringed musical instruments can be classified into three types – plucked, struck and bowed. Guiter, mandoline, sitar, tanpura, sarode, vina etc. are the plucked type instruments. Piano and santur are struck type instruments whereas violin, esraj, sarengi etc. are bowed type instruments. In plucked and bowed type instruments the amplitude of the sth harmonic is proportional to $\frac{1}{s^2}$ where as in struck type instruments it is proportional to $\frac{1}{s}$. So more number of harmonics will have appreciable amplitude in the case of struck string. Hence sound emitted will be richer in harmonics.

Further reading:

(1). Principles of acoustics, Basudev Ghosh
(2). Sound, K. Bhattacharyya
(3). Waves and Oscillations, R. N. Chaudhuri