**Fourier's Theorem and its application for some waveforms**

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# 1 Fourier's theorem

## 1.1 Statement

Any finite complex periodic motion may be regarded as the sum total of an infinite number of simple harmonic motions of commensurate period, i.e. the displecement for the periodic motion at any time $t$ may be expressed as the sum of an infinite number of sine and cosine terms.
Mathematically, the instantaneous displacement may be written as

$$\begin{aligned} y &= A_0 + (A_1 \cos\omega t + A_2 \cos 2\omega t + A_3 \cos 3\omega t + .......) \\ &+ (B_1 \sin\omega t + B_2 \sin 2\omega t + B_3 \sin 3\omega t + .....) \end{aligned}$$

where $A_0$, $A_1$, $A_2$, $B_1$, $B_2$ are arbitrary constants.

$$\therefore y = A_0 + \sum_{s=1}^{\infty} A_s \cos s\omega t + \sum_{s=1}^{\infty} B_s \sin s\omega t \tag{1}$$

where $A_0$, $A_s$'s , $B_s$'s are Fourier constants.

## 1.2 Evaluation of Fourier Constants:

(i) $A_0$ :
In order to evaluate $A_0$, we multiply both sides of equation (1) by $dt$ and integrate the results with respect to time $t$ over the complete period $T$.

$$\therefore \int_0^T y dt = \int_0^T A_0 dt + \int_0^T \sum_{s=1}^{\infty} A_s \cos s\omega t dt + \sum_{s=1}^{\infty} B_s \sin s\omega t dt \tag{2}$$

Now

$$\begin{aligned} \int_0^T \sum_{s=1}^{\infty} A_s \cos s\omega t dt &= \left[\frac{A_s \sin s\omega t}{s\omega}\right]_0^T \\ &= \frac{As}{s\omega}\left[\sin s\omega t - \sin 0\right] \\ &= \frac{As}{s\omega}\left[\sin 2\pi s - \sin 0\right] = 0 \end{aligned}$$

$$\begin{aligned} \int_0^T \sum_{s=1}^{\infty} B_s \sin s\omega t dt &= \left[\frac{B_s \cos s\omega t}{s\omega}\right]_T^0 \\ &= \frac{Bs}{s\omega}\left[\cos 0 - \cos s\omega t\right] \\ &= \frac{Bs}{s\omega}\left[\cos 0 - \cos 2\pi s\right] \\ &= \frac{Bs}{s\omega}\left[1 - 1\right] = 0 \end{aligned}$$

From (2)

$$\int_0^T y dt = A_0 T$$

$$A_0 = \frac{1}{T}\int_0^T y dt$$

(ii) $A_s$ :

In order to evaluate $A_s$ we multiply both sides of equation (1) by $\cos m\omega t$ (where m is an integer) and integrate the results with respect to $t$ over a complete period.

$$\begin{aligned} \int_0^T y \cos m\omega t dt &= \int_0^T A_0 \cos m\omega t dt \\ &+ \int_0^T \left( \sum_{s=1}^{\infty} A_s \cos s\omega t \right) \cos m\omega t dt \\ &+ \int_0^T \left( \sum_{s=1}^{\infty} B_s \sin s\omega t \right) \cos m\omega t dt \end{aligned} \tag{3}$$

Now

$$\begin{aligned} \int_0^T A_s \cos s\omega t \cos m\omega t dt &= \frac{A_s}{2}\int_0^T \left[\cos(s+m)\omega t + \cos(s-m)\omega t\right] dt \\ &= \frac{A_s}{2}\left[\frac{\sin(s+m)\omega t}{(s+m)\omega} + \frac{\sin(s-m)\omega t}{(s-m)\omega}\right]_0^T \\ &= 0 \ when \ s \neq m \\ &= A_m \frac{T}{2} \ when \ s = m. \end{aligned}$$

$$\begin{aligned} \int_0^T B_s \sin s\omega t \cos m\omega t dt &= \frac{B_s}{2}\int_0^T \left[\sin(s+m)\omega t + \sin(s-m)\omega t\right] dt \\ &= \frac{B_s}{2}\left[\frac{\cos(s+m)\omega t}{(s+m)\omega} + \frac{\cos(s-m)\omega t}{(s-m)\omega}\right]_T^0 \\ &= 0 \ when \ s = m \ or \ s \neq m. \end{aligned}$$

From (3)

$$\int_0^T y \cos m\omega t dt = A_m \frac{T}{2}$$

$$A_m = \frac{2}{T}\int_0^T y \cos m\omega t dt$$

$$A_s = \frac{2}{T}\int_0^T y \cos s\omega t dt$$

(iii) $B_s$ :
In order to evaluate $B_s$, we multiply both sides of equation (2) by $\sin m\omega t$ and integrate the results over a complete period. Hence

$$\begin{aligned}
\int_0^T y \sin m\omega t dt &= \int_0^T A_0 \sin m\omega t dt \\
&+ \int_0^T \left( \sum_{s=1}^{\infty} A_s \cos s\omega t \right) \sin m\omega t dt \\
&+ \int_0^T \left( \sum_{s=1}^{\infty} B_s \sin s\omega t \right) \sin m\omega t dt
\end{aligned} \tag{4}$$

Proceeding similarly as done in the above case, we have

$$\int_0^T y \sin m\omega t dt = 0 + B_m \frac{T}{2}$$

$$B_m = \frac{2}{T} \int_0^T y \sin m\omega t dt$$

$$B_s = \frac{2}{T} \int_0^T y \sin s\omega t dt$$

# 2 Applications

## 2.1 Square wave

A particle undergoes a periodic motion in such a way that its displacement ($y$) is given by

$$\begin{aligned}
y &= 0 \; for \; 0 < t < \frac{T}{2} \\
y &= k = \; const \; for \; \frac{T}{2} < t < T.
\end{aligned}$$

Express $y$ as a fourier series.

**Solution:**
By Fourier theorem, the displacement of the particle at time $t$ may be expressed as

$$y = A_0 + \sum_{s=1}^{\infty} A_s \cos s\omega t + \sum_{s=1}^{\infty} B_s \sin s\omega t \tag{5}$$

where $A_0$, $A_s$'s , $B_s$'s are Fourier constants.

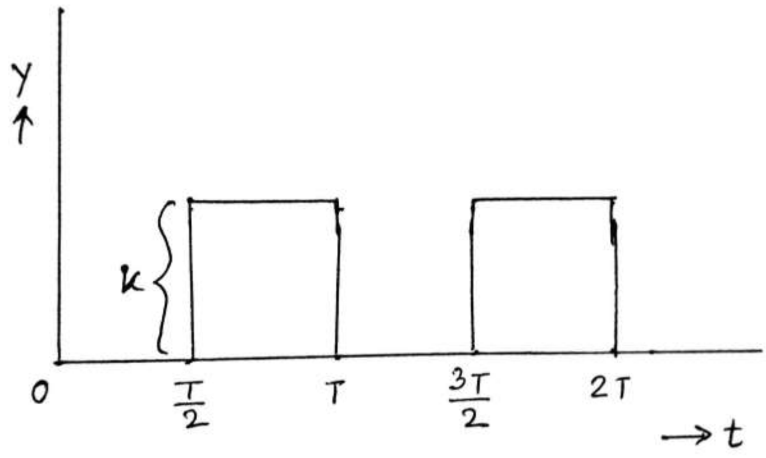


Figure 1: *Square wave*

Now

$$
\begin{aligned}
A_0 &= \frac{1}{T}\int_0^T y dt \\
&= \frac{1}{T}\left[\int_0^{T/2} 0.dt + \int_{T/2}^{T} k dt\right] \\
&= \frac{k}{2}
\end{aligned}
$$

Again

$$
\begin{aligned}
A_s &= \frac{2}{T}\int_0^T y \cos s\omega t dt \\
&= \frac{2}{T}\int_{T/2}^{T} k \cos s\omega t dt \\
&= \frac{2k}{T}\left[\frac{\sin s\omega t}{s\omega}\right]_{T/2}^{T} \\
&= \frac{2k}{s\omega T}\left[\sin 2\pi s - \sin \pi s\right] \\
&= 0
\end{aligned}
$$

Lastly

$$
\begin{aligned}
B_s &= \frac{2}{T}\int_0^T y \sin s\omega t dt \\
&= \frac{2}{T}\int_{T/2}^{T} k \sin s\omega t dt \\
&= \frac{2k}{T}\left[\frac{\cos s\omega t}{s\omega}\right]_T^{T/2} \\
&= \frac{2k}{s\omega T}\left[\cos \pi s - \cos 2\pi s\right] \\
&= \frac{2k}{s.2\pi}[-1-1] \; when \; s \; is \; odd \\
&= 0 \; when \; s \; is \; even
\end{aligned}
$$

$$
\therefore B_s = \frac{-2k}{\pi s},
$$

where $s$ is only odd. Hence, the required Fourier series as

$$
y = \frac{k}{2} - \frac{2k}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + .......\right) \tag{6}
$$

$\Rightarrow$ only odd harmonics are present in the vibration.

## 2.2 Saw tooth wave

A particle undergoes a periodic motion in such a way that its displacement rises to a maximum value monotonically with respect to time and after the completion of one period it suddenly comes to the original position. EXpress the general displacement of the particle in a fourier series.

**Solution:**

The time displacement curve for the particle is shown in Figure 2. As evident from the diagram, the general displacement $y$ at any time $t$ may be obtained as

$$
\begin{aligned}
\frac{y}{k} &= \frac{t}{T} \\
or \; y &= \frac{kt}{T}
\end{aligned} \tag{7}
$$

The general displacement may be expressed as a fourier series like

$$
y = A_0 + \sum_{s=1}^{\infty} A_s \cos s\omega t + \sum_{s=1}^{\infty} B_s \sin s\omega t \tag{8}
$$

where $A_0$, $A_s$'s , $B_s$'s are Fourier constants.

Now

$$
\begin{aligned}
A_0 &= \frac{1}{T}\int_0^T y dt \\
&= \frac{1}{T}\frac{k}{T}\int_0^T t dt \\
&= \frac{k}{2}
\end{aligned}
$$

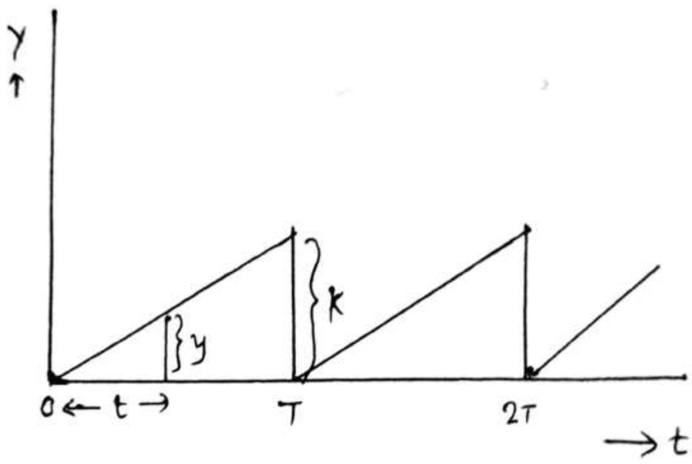


Figure 2: *Saw tooth wave*

Again

$$\begin{aligned}
A_s &= \frac{2}{T}\int_0^T y\cos s\omega t dt \\
&= \frac{2}{T}\frac{k}{T}\int_0^T t\cos s\omega t dt \\
&= \frac{2k}{T^2}\left[t\frac{\sin s\omega t}{s\omega} + \frac{\cos s\omega t}{s^2\omega^2}\right]_0^T \\
&= \frac{2k}{T^2}\left[\frac{1}{s^2\omega^2} - \frac{1}{s^2\omega^2}\right] \\
&= 0
\end{aligned}$$

Lastly

$$\begin{aligned}
B_s &= \frac{2}{T}\int_0^T y\sin s\omega t dt \\
&= \frac{2k}{T^2}\int_0^T t\sin s\omega t dt \\
&= \frac{2k}{T^2}\left[\frac{-t\cos s\omega t}{s\omega} + \frac{\sin s\omega t}{s^2\pi^2}\right]_0^T \\
&= \frac{2k}{T^2}[-\frac{T}{s\omega}] \quad [\because \omega T = 2\pi]
\end{aligned}$$

$$\therefore B_s = \frac{-2k}{s\omega t} = \frac{-k}{\pi s}$$

where $s$ is only odd. Hence, the required Fourier series as

$$y = \frac{k}{2} - \frac{k}{\pi}\left(\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + .......\right) \tag{9}$$

⇒ all the harmonics are present.