Superposition of vibrations -II

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1 Superposition of two perpendicular Harmonic Oscillations

1.1 Lissajous' figures

The figures formed by two vibrations at right angles to each other are known as Lissajous' Figures. The figures are of importance in sound. With their help, the equality of , or a slight difference between the frequencies of two sounding bodies can be tested.

1.2 Oscillations having same frequencies

Let the vibration be

$$\begin{aligned} x &= a\cos(\omega t + \delta_1) \quad (1) \\ y &= b\cos(\omega t + \delta_2) \quad (2) \end{aligned}$$

From equation 2 we can write

$$\frac{y}{b} = \cos(\omega t + \delta_1 + \delta_2 - \delta_1) = \cos(\omega t + \delta_1 + \delta)$$

where$\delta = \delta_2 - \delta_1$ is the phase difference between the two SHMs. Therefore,

$$\frac{y}{b} = \cos(\omega t + \delta_1)\cos\delta - \sin(\omega t + \delta_1).\sin\delta$$

Using equation 1 we can write

$$\frac{y}{b} = \frac{x}{a}\cos\delta - \sqrt{1 - \frac{x^2}{a^2}}\sin\delta$$

or

$$\left(\frac{x}{a}\cos\delta - \frac{y}{b}\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\delta$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta = \sin^2\delta \quad (3)$$

It represents the general equation of an ellipse bounded within a rectangle of sides $2a$ and $2b$. The major axis of the ellipse makes an angle θ with the $x-$ axis, which is given by

$$\tan 2\theta = \frac{2ab}{a^2 - b^2}\cos\delta \quad (4)$$

Thus the motion in general is elliptical. Let us consider a few special cases:

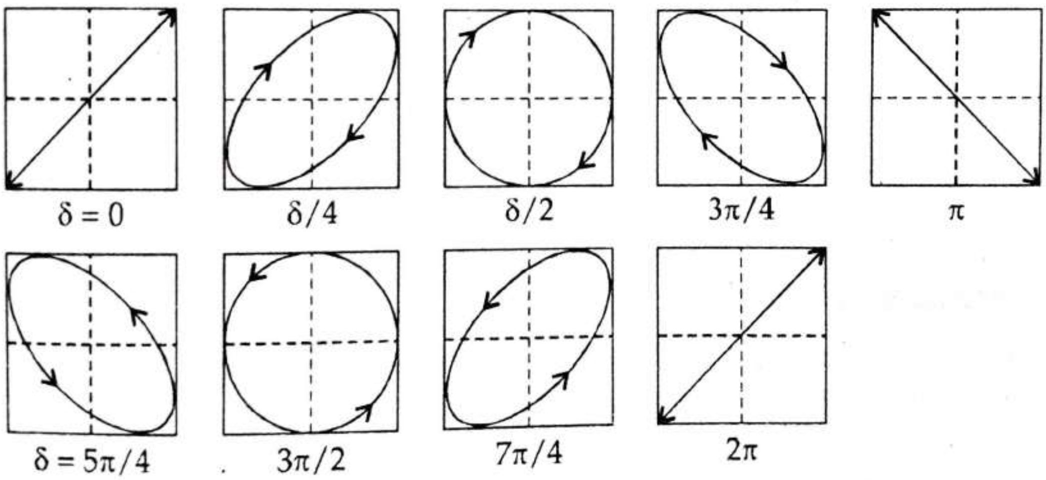

Figure 1: *Resultant pattern due to superposition of two rectangular SHMs*

(i) If $\delta = \delta_2 - \delta_1 = 0$, i.e., the two SHMs are in phase, then the equation 3 reduces to

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} &= 0 \\ or, \left(\frac{x}{a} - \frac{y}{b}\right)^2 &= 0 \end{aligned}$$

It represents a pair of coincident straight lines $y = \frac{b}{a}x$, passing through the origin and inclined to the $x-$ axis at an angle $\tan^{-1}(\frac{b}{a})$.

(ii). When $\delta = \pi$, equation 3 becomes

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

This also represents a pair of coincident straight lines passing through the origin inclined to the $x-$ axis at an angle θ given by $\tan\theta = -\frac{b}{a}$.

(iii). When $\delta = \pi/2$, equation 3 becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It represents an ellipse with semi-axes a and b along the coordinate axes. If in addition $a = b$ the ellipse degenerates into a circle.

$$x^2 + y^2 = a^2$$

The direction of motion in the elliptic or circular path can be ascertained from the equations defining the component motions. Let us consider two rectangular SHMs of same frequency but differing in phase by $\pi/2$.

$$\begin{aligned} x &= a\cos\omega t \\ y &= b\cos(\omega t + \pi/2) = -b\sin\omega t \end{aligned}$$

These two combine to give an elliptic motion. At time $t = 0$ the position of the point $P(x, y)$ is at $(a, 0)$. Now as t increases x decreases from its maximum value a and y begins to go negative. This indicates that the point P moves in clockwise sense.

Similarly the equations

$$\begin{aligned} x &= a\cos\omega t \\ y &= b\cos(\omega t + 3\pi/2) = b\sin\omega t \end{aligned}$$

represent an elliptic motion in anticlockwise sense.

2 Oscillations having frequency ratio $1:2$

Let the two vibrations of frequencies in the ratio 1:2 differening in phases by δ be imposed on a particle. Then we can write the component displacement as

$$x = a\cos\omega t \tag{5}$$
$$y = b\cos(2\omega t + \delta) \tag{6}$$

$$\begin{aligned} \therefore \frac{y}{b} &= \cos 2\omega t.\cos\delta - \sin 2\omega t.\sin\delta \\ &= (2\cos^2\omega t - 1)\cos\delta - 2\sin\omega t.\cos\omega t.\sin\delta \end{aligned}$$

Using equation 5 we get

$$\frac{y}{b} = \left(2\frac{x^2}{a^2} - 1\right)\cos\delta - 2\frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\sin\delta$$

$$\left[\frac{y}{b} + \cos\delta - \frac{2x^2}{a^2}\cos\delta\right]^2 = \frac{4x^2}{a^2}\left(1 - \frac{x^2}{a^2}\right)\sin^2\delta \tag{7}$$

This is an equation of fourth degree in x and, in general, represents closed curce having two loops. For a given value of δ the exact nature of the curve can be tracr=ed. For example, if $\delta = 0$ then equation 7 reduces to

$$\left(\frac{y}{b} + 1 - \frac{2x^2}{a^2}\right)^2 = 0$$

It represents two coincident parabolas (Figure 2 (a)) given by

$$x^2 = \frac{a^2}{2b}(y + b)$$

If $\delta = \pi/2$ equation 7 reduces to

$$\frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1\right) + \frac{y^2}{b^2} = 0$$

This equation represents a curve containing two loops as shown in Figure 2(c).

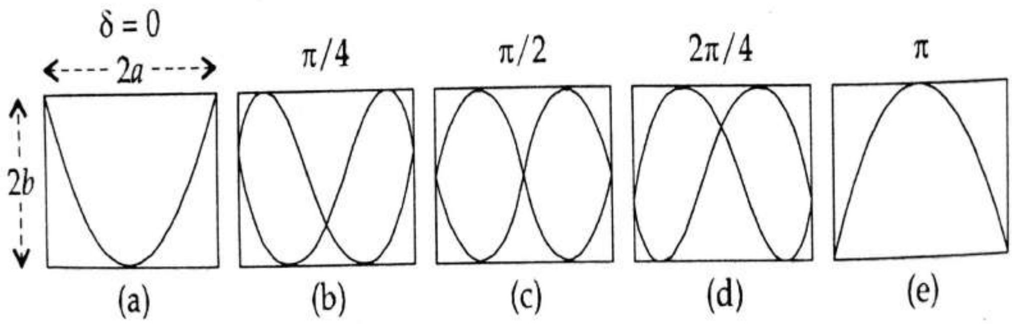

Figure 2: *Resultant pattern of two rectangular SHMs having frequency ratio 1:2*

2.1 Frequency ratio 1:3

If the frequencies of the vibrations are in the ratio 1 : 3, then we can write, when the vibrations differ in phase by δ,

$$\begin{aligned} x &= a \sin \omega t \\ y &= b \sin(3\omega t + \delta) \end{aligned}$$

We have

$$\frac{y}{b} = (3 \sin \omega t - 4 \sin^3 \omega t) \cos \delta + (4 \cos^3 \omega t - 3 cos\omega t) \sin \delta$$

or

$$\left[\frac{y}{b} - \left(\frac{3x}{a} - \frac{4x^3}{a^3}\right) cos\delta\right]^2 = \left(1 - \frac{x^2}{a^2}\right) \left[4\left(1 - \frac{x^2}{a^2}\right) - 3\right]^2 \sin^2 \delta \tag{8}$$

This gives the general equation of the resultant motion for any phase difference δ and amplitudes a and b.

Special cases: (i). When $\delta = 0$ equation 8 reduces to

$$\left[\frac{y}{b} - \left(\frac{3x}{a} - \frac{4x^3}{a^3}\right)\right]^2 = 0 \tag{9}$$

which gives two coincident cubic curves.
(ii). When $\delta = \pi/2$, equation 8 becomes

$$\frac{y^2}{b^2} = \left(1 - \frac{x^2}{a^2}\right) \left(1 - \frac{4x^2}{a^2}\right)^2 \tag{10}$$

This is an equation of the sixth degree giving an orbit of three loops. In general if the ratio of the frequencies is N, the curve will have N loops.

2.2 Assignment

1. A particle is subjected to two mutually perpendicular simple harmonic oscillations,

$$
\begin{aligned}
x &= 2\cos t \\
y &= cos(t+4)
\end{aligned}
$$

Trace the trajectory of the particle using graphical method.

2. Determine the shape of the Lissajous' figure for the resultant motion, if a particle is subjected to the following SHMs:

$$
\begin{aligned}
x &= 2\sin 2\pi t \\
y &= 3\sin \pi t
\end{aligned}
$$

Books Suggested:
(1). Principles of acoustics, Basudev Ghosh
(2). Sound, K. Bhattacharyya
(3). Waves and Oscillations, R. N. Chaudhuri