Wave equation of a stretched string

Dr. Soma Mandal,
Assistant professor,
Department of Physics, Government Girls' General Degree College, Kolkata
Paper code:PHS-G-CC-4-4-TH

1 Wave equation in stretched string and its solutions

We consider a stretched string of lebgth l rigidly fixed at its end points. In the undisturbed orientation the string lies along $X-$ axis. If this string be excited to transverse vibration, transverse wave will be generated from the point of excitation.

We choose an elementay portion AB of string of length δx . In course of vibration let the configuration of this portion at any time t be $A'B'$, such that the displacement of the point A is y and that of B is $y + \delta y$. If the displacement be small the tension T acting on the string will remain constant.

For the displaced portion $A'B'$ of the string, the tension of A' acts along the tangent $A'M$ while that at the point B' acts along the tangent NB'. Thus the net force acting on the element $A'B'$ along Y direction at time t is

$$\begin{aligned} T(\sin\theta_2 - \sin\theta_1) &= T(\tan\theta_2 - \tan\theta_1) \qquad [As\ \theta_1\ and\ \theta_2\ are\ each\ small] \\ &= T\left[\frac{\partial}{\partial x}(y+\delta y) - \frac{\partial y}{\partial x}\right] \\ &= T\left[\frac{\partial}{\partial x}(y + \frac{\partial y}{\partial x}\delta x) - \frac{\partial y}{\partial x}\right] \\ &= T\frac{\partial^2 y}{\partial x^2}\delta x \end{aligned} \tag{1}$$

If m be the mass per unit length of the string and $\frac{\partial^2 y}{\partial x^2}$ represents the acceleration of the portion of the string at time t, we must have

$$T\frac{\partial^2 y}{\partial x^2}\delta x = m\delta x\frac{\partial^2 y}{\partial t^2}$$

where $m\delta x$ is the total mass of the element δx. or

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

or

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} \tag{2}$$

where $v = \sqrt{\frac{T}{m}}$= velocity of propagation of transverse waves along the length of the string. The general solution of equation 2 is given by

$$y = f_1(vt - x) + f_2(vt + x) \tag{3}$$

where $f_1(vt - x)$ represents a wave proceeding along $+ve$ x direction, while $f_2(vt - x)$ denotes another wave travelling along $-ve$ x direction.

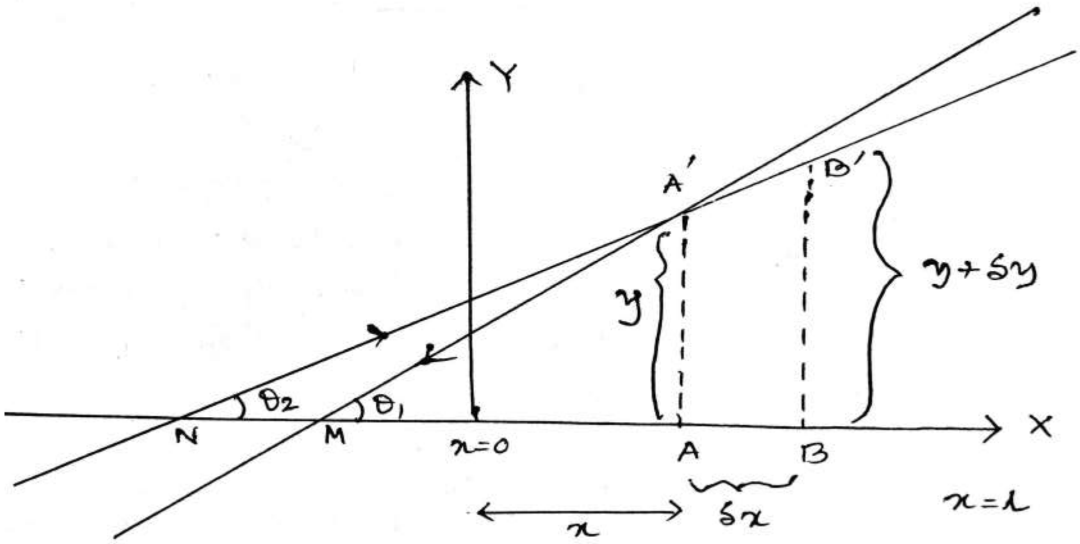

Figure 1: *Stretched string*

2 Production of stationary waves on string

We have from equation 2 the general wave equation correspond to transverse vibration of the stretched string as

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

Let

$$y(x,t) = \phi(x)\theta(t) \tag{4}$$

where $\phi(x)$ is exclusively a function of x and $\theta(t)$ is entirely a function of t.

$$\frac{\partial^2 y}{\partial t^2} = \phi(x)\frac{\partial^2 \theta}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \theta(t)\frac{\partial^2 \phi}{\partial x^2}$$

From equation 2

$$\phi(x)\frac{\partial^2 \theta}{\partial t^2} = v^2 \theta(t)\frac{\partial^2 \phi}{\partial x^2}$$

or

$$\frac{1}{\theta}\frac{\partial^2 \theta}{\partial t^2} = \frac{v^2}{\phi}\frac{\partial^2 \phi}{\partial x^2} \tag{5}$$

The L.H.S of the equation is exclusively a function of t while the R.H.S is entirely a function of x. Hence, each side must be a constant. The constant is real, and for the finite bounded vibration of the string this constant must be $-ve$. We choose it as ω^2. Thus from equation 5

$$\frac{\partial^2\theta}{\partial t^2} + \omega^2\theta = 0 \tag{6}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\omega^2}{v^2}\phi = 0 \tag{7}$$

The general solution of 6 is given by

$$\theta(t) = A\cos\omega t + B\sin\omega t \tag{8}$$

and that of 7 is given by

$$\phi(x) = C\cos\frac{\omega x}{v} + D\sin\frac{\omega x}{v} \tag{9}$$

where A, B, C and D are arbitrary constants. Thus, the general solution of equation 2 is written as

$$y = (A\cos\omega t + B\sin\omega t)\left(C\cos\frac{\omega x}{v} + D\sin\frac{\omega x}{v}\right) \tag{10}$$

Since $y = 0$ both at $x = 0$ and $x = l$ (l=length of the string) for all time t, we have from 10

$$0 = (A\cos\omega t + B\sin\omega t)\,C \Rightarrow C = 0$$

$$\therefore y = (A\cos\omega t + B\sin\omega t)D\sin\frac{\omega l}{v} \tag{11}$$

$$\Rightarrow \sin\frac{\omega l}{v} = 0, \because D \neq 0$$

Since, in this case $\phi(x)$ becomes trivial

$$\begin{aligned} \therefore \sin\frac{\omega l}{v} &= 0 \\ or\ \frac{\omega l}{v} &= s\pi\ where\ s = 1, 2, 3.... \\ or\ \omega &= \frac{s\pi v}{l} \end{aligned} \tag{12}$$

Hence from equation 11

$$\begin{aligned} y &= (A\cos\frac{s\pi vt}{l} + B\sin\frac{s\pi vt}{l})D\sin\frac{s\pi x}{l} \\ y &= (A_s\cos\frac{s\pi vt}{l} + B_s\sin\frac{s\pi vt}{l})\sin\frac{s\pi x}{l} \end{aligned} \tag{13}$$

where $A_s = A.D$ and $B_s = B.D$ But

$$\omega = \frac{s\pi}{l}\sqrt{\frac{T}{m}} \qquad (\because v = \sqrt{\frac{T}{m}}) \tag{14}$$

$$\therefore \frac{\omega}{2\pi} = f_s = \frac{s}{2l}\sqrt{\frac{T}{m}} \tag{15}$$

This f_s represent the characteristic frequencies of the s th mode of vibration of the string. Hence the frequency of the fundamental tone is

$$f_1 = \frac{1}{2l}\sqrt{\frac{T}{m}} \tag{16}$$

Since equation (2) is a linear homogeneous equation and equation (13) represents a solution of equation (2) corresponding to the s th mode of vibration, the sum solution for all possible values of s will again be a solution of (2). Thus the complete solution of equation (2) is given by

$$y = \sum_{s=1}^{\infty}(A_s \cos\frac{s\pi vt}{l} + B_s \sin\frac{s\pi vt}{l}) \sin\frac{s\pi x}{l} \tag{17}$$