More on matrices

Soma Mandal

Department of Physics,
Government Girls' General Degree College

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Conjugate of a Matrix

Let A be a $m \times n$ matrix over $\mathbb{C}$. If $A = [a_{ij}]$ then the conjugate of A, denoted by $\bar{A}$, is the matrix B $B = [b_{ij}]$ with $b_{ij} = \bar{a}_{ij}$.

- Let A= $\begin{bmatrix} 1 & 4+3i & i \\ 0 & 1 & i-2 \end{bmatrix}$. Then $\bar{A} = \begin{bmatrix} 1 & 4-3i & -i \\ 0 & 1 & -i-2 \end{bmatrix}$
- Let A= $\begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$. Then $\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$

Hermitian Matrix

- A square matrix $A = [a_{ij}]$ is called Hermitian Matrix, if every i-j th element of A is equal to conjugate complex j-i th element of A.
- In other words $a_{ij} = \bar{a}_{ji}$.

Example: $\begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix}$

- Hence all the elements of the principal diagonal are real.
- A necessary and sufficient condition for a matrix A to be Hermitian is that $A = A^{\theta}$

Hermitian Matrix: Example

- Prove the following:
  - (i) $(A^\theta)^\theta = A$
  - (ii) $(A+B)^\theta = A^\theta + B^\theta$
  - (iii) $(kA)^\theta = \bar{k}A^\theta$
  - (iv) $(AB)^\theta = B^\theta \cdot A^\theta$
- Solution:
  - (i) $(A^\theta)^\theta = \overline{[\{\overline{(A)}'\}']} = [A]$
  - (ii) $(A+B)^\theta = (\overline{A+B})' = (\overline{A}+\overline{B})' = (\overline{A})' + (\overline{B})' = A^\theta + B^\theta$
  - (iii)$(kA)^\theta = (\overline{kA})' = (\overline{k}\overline{A})' = \overline{k}(\overline{A})' = \overline{k}A^\theta$
  - $(AB)^\theta = (\overline{AB})' = (\overline{A}\cdot\overline{B})' = (\overline{B})' \cdot (\overline{A})' = B^\theta \cdot A^\theta$

Hermitian Matrix: Example

- Prove the matrix $A = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$ is Hermitian.
  - Solution: $\overline{A} = \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & -i \\ 2 & i & 0 \end{bmatrix}$

    and $\overline{A}' = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$

    $\implies A^{\theta} = A$. Hence A is Hermitian matrix.

Skew-Hermitian Matrix

- A square matrix $A = [a_{ij}]$ will be called a Skew Hermitian Matrix, if every i-j th element of A is equal to negative conjugate complex j-i th element of A.
- In other words $a_{ij} = -\bar{a}_{ji}$. All the elements in the principal diagonal will be of the form $a_{ij} = -\bar{a}_{ji}$
  $a_{ij} + \bar{a}_{ji} = 0$.
  If a_{ii}=$a + ib$
  then $\bar{a}_{ii} = a - ib$
  Therefore $(a + ib) + (a - ib) = 0$ or $a = 0$
  So a_{ii} is pure imaginary or $a_{ii} = 0$.

Example: $\begin{bmatrix} i & 2-3i & 4+5i \\ -(2+3i) & 0 & 2i \\ -(4-5i) & 2i & -3i \end{bmatrix}$

Skew-Hermitian Matrix

- Hence all the diagonal elements of a Skew Hermitian Matrix are either zeros or pure imaginary.
- The necessary and sufficient condition for a matrix A to be Skew Hermitian is that
  - $A^\theta = -A$
  - $(\bar{A})' = -A$
- Where A^θ is the transpose of the conjugate of a matrix A.

Skew-Hermitian Matrix: example

- Show that $A = \begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$ is skew-Hermitian matrix.
- Solution: $\overline{A} = \begin{bmatrix} i & 3-2i & -2+i \\ -3-2i & 0 & 3+4i \\ 2+i & -3+4i & 2i \end{bmatrix}$

$(\overline{A})' = \begin{bmatrix} i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix}$

$\because A^\theta = (\overline{A})'$

$A^\theta = \begin{bmatrix} i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix} =$

$- \begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix} = -A$

$A^\theta = -A \Longrightarrow$ is skew-Hermitian matrix.

Singular Matrix and non-singular Matrix

- If the determinant of the matrix is zero, then the matrix is known singular matrix.
- Example: If $|A| = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, then A is a singular matrices.
- A non-singular matrix is a square one whose determinant is not zero.
- The rank of a matrix $[A]$ is equal to the order of the largest non-singular submatrix of $[A]$. It follows that a non-singular square matrix of $n \times n$ has a rank of n. Thus, a non-singular matrix is also known as a full rank matrix. For a non-square $[A]$ of $m \times n$, where $m > n$, full rank means only n columns are independent.

Trace of a Matrix

The trace of a square matrix is the sum of its diagonal elements.

- Definition: Let A be a $n \times n$ matrix. Then, its trace, denoted by $trace(A)$ or $tr(A)$, is the sum of its diagonal elements.

$$trA = \sum_{n=1}^{n} A_{nn}$$

Example: Define the matrix $A = \begin{bmatrix} 2 & 1 & 5 \\ 2 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix}$ Then, its trace is

$$tr(A) = A_{11} + A_{22} + A_{33} = 2 + 3 + 0 = 5$$

Properties

- Trace of a sum: The trace of a sum of two matrices is equal to the sum of their trace.
- Proposition: Let A and B be two $n \times n$ matrices. Then

$$tr(A+B) = tr(A) + tr(B)$$

- Proof:

$$\begin{aligned} tr(A+B) &= \sum_{n=1}^{n} (A+B)_{nn} \\ &= \sum_{n=1}^{n} (A_{nn} + B_{nn}) \\ &= \sum_{n=1}^{n} A_{nn} + \sum_{n=1}^{n} B_{nn} \\ &= tr(a) + tr(B) \end{aligned}$$

Properties of trace of a matrix

- Trace of a scalar multiple:
  Let A be a $n \times n$ matrix and α a scalar.
  Then $tr(\alpha A) = \alpha tr(A)$
- Proof:

$$
\begin{aligned}
tr(\alpha A) &= \sum_{n=1}^{n} (\alpha A)_{nn} \\
&= \sum_{n=1}^{n} \alpha A_{nn} \\
&= \alpha \sum_{n=1}^{n} A_{nn} \\
&= \alpha tr(A)
\end{aligned}
$$

Some other properties

- Let A and B be two $K \times K$ matrices and α and β two scalars. $tr(\alpha A + \beta B) = \alpha tr(A) + \beta tr(B)$
- $tr(A^T) = tr(A)$
- Let A be a $K \times L$ matrix and B be an $L \times K$ matrix. Then $tr(AB) = tr(BA)$
- Assignment: Prove the above relations.

Exercise:

- Write matrix (A) given below as the sum of a symmetric and a skew symmetric matrix. $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$
- If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$, obtain the product AB and explain why BA is not defined.
- If If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ Verify that $(AB)C = A(BC)$ and $A(B + C) = AB + AC$.
- Determine the values of α, β, γ when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.
- For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular?

Books Suggested:

- Mathematical Methods for Physicists: A comprehensive Guide; ARFKEN, WEBER, and HARRIS, 7th Edition.
- Advanced Engineering mathematics; H.K. Dass
- Taboga, Marco (2017). "Trace of a matrix", Lectures on matrix algebra.