Adjoint and Inverse of a Matrix

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Adjoint of a square Matrix

Let the determinant of the square matrix A be $\mid A \mid$.

- If A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. Then $\mid A \mid = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- The matrix formed by the co-factors of the element in $\mid A \mid$ is

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$$

Adjoint of a square matrix

- $A_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2 c_3 - b_3 c_2$
- $A_3 = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1 c_2 - b_2 c_1$
- $B_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1 c_3 - a_3 c_1$
- $C_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} = a_2 b_3 - a_3 b_2$
- $C_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

- $A_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -b_1 c_3 + b_3 c_1$
- $B_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} = -a_2 c_3 + a_3 c_2$
- $B_3 = -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = -a_1 c_2 + a_2 c_1$
- $C_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = -a_1 b_3 + a_3 b_1$

Then the transpose of the matrix of the cofactors

$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$ is called the adjoint of the matrix A and is written as $adj\ A$.

Property of adjoint matrix

The product of a matrix A and its adjoint is equal to unit matrix multiplied by the determinant.

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ and $adj\ A = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$.

$$A \cdot (adj\ A) = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \times \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 A_1 + a_2 A_2 + a_3 A_3 & a_1 B_1 + a_2 B_2 + a_3 B_3 & a_1 C_1 + a_2 C_2 + a_3 C_3 \\ b_1 A_1 + b_2 A_2 + b_3 A_3 & b_1 B_1 + b_2 B_2 + b_3 B_3 & b_1 C_1 + b_2 C_2 + b_3 C_3 \\ c_1 A_1 + c_2 A_2 + c_3 A_3 & c_1 B_1 + c_2 B_2 + c_3 B_3 & c_1 C_1 + c_2 C_2 + c_3 C_3 \end{bmatrix}$$

$$\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\ I$$

Inverse of a Matrix

- If A and B are two square matrices of the same order, such that $AB = BA = I$ then B is called the inverse of A i.e. $B = A^{-1}$ and A is the inverse of B.
- Condition for a square matrix A to possess an inverse is that matrix A is non-singular, i.e. $\mid A \mid \neq 0$.
  If A is a square matrix ans B be its inverse, then $AB = I$.
  Taking determinant of both sides $\mid AB \mid = \mid I \mid$ or $\mid A \mid\mid B \mid = I$.
  From this relation it is clear that $\mid A \mid \neq 0$
  i.e. the matrix A is non-singular.

Inverse of a matrix

To find the inverse matrix with the help of adjoint matrix

- we know that $A \cdot adj(A) = \mid A \mid I$

$$\frac{A \cdot adj(A)}{\mid A \mid} = I$$

  provided $\mid A \mid \neq 0$.

- $AA^{-}1 = I$

$$\therefore A^{-1} = \frac{1}{\mid A \mid} adj(A)$$

Solved problems

- if $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}.
- Solution: $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

  $\mid A \mid = 3(-3+4) + 3(2-0) + 4(-2-0) = 3 + 6 - 8 = 1.$

  The cofactor of elements of various rows of $\mid A \mid$ are

  $\begin{bmatrix} (-3+4) & (-2-0) & (-2-0) \\ (3-4) & (3-0) & (3-0) \\ (-12+12) & (-12+8) & (-9+6) \end{bmatrix}$

- The cofactor of elements of various rows of $\mid A \mid$ are
$$\begin{bmatrix} (-3+4) & (-2-0) & (-2-0) \\ (3-4) & (3-0) & (3-0) \\ (-12+12) & (-12+8) & (-9+6) \end{bmatrix}$$
Therefore the matrix formed by the co-factor of $\mid A \mid$ is
$$\begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$
$$adj(A) = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$
$$\therefore A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{1} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Solved problems

- Prove that the inverse of a matrix is unique.

  We suppose that B and C are two inverse matrices of a given matrix A.
  Then $AB = BA = I \because B$ is inverse of A.
  and $AC = CA = I \because C$ is inverse of A.
  But $C \cdot (AB) = (CA) \cdot B$
  $\implies C \cdot I = I \cdot B$ or $C = B$
  Hence, the inverse of matrix A is unique.

Exercise:

- Find the adjoint of the following matrices (i) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$
- If If $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ show that $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Find the condition of k such that the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & k & 6 \\ -1 & 5 & 1 \end{bmatrix}$ has an inverse. Obtain $A^{-1}$for $k = 1$
- Prove that $(A^{-1})^T = (A^T)^{-1}$.

Books Suggested:

- Mathematical Methods for Physicists: A comprehensive Guide; ARFKEN, WEBER, and HARRIS, 7th Edition.
- Advanced Engineering mathematics; H.K. Dass