University of Calcutta
Semester 2
PHYSICS
Paper: PHS-G-CC-2-2-TH (NEW SYLLABUS)

GRADIANT, DIVERGENCE, CURL OF VECTOR
SOLVED PROBLEMS
ASSIGNMENTS

Dr. Koel Adhikary,
Department of Physics
Government Girls' General Degree College

Vector Analysis

2.14 The Vector Differential Operator $\vec{\nabla}$

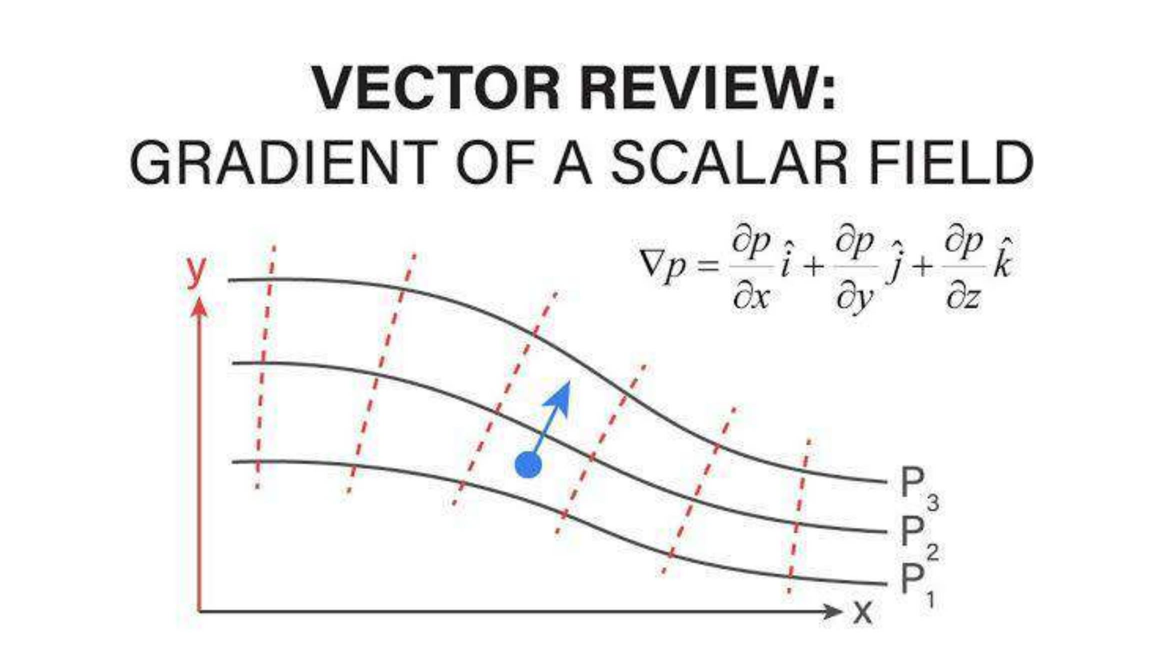
The differential operator $\vec{\nabla}$ is defined in Cartesian notation by

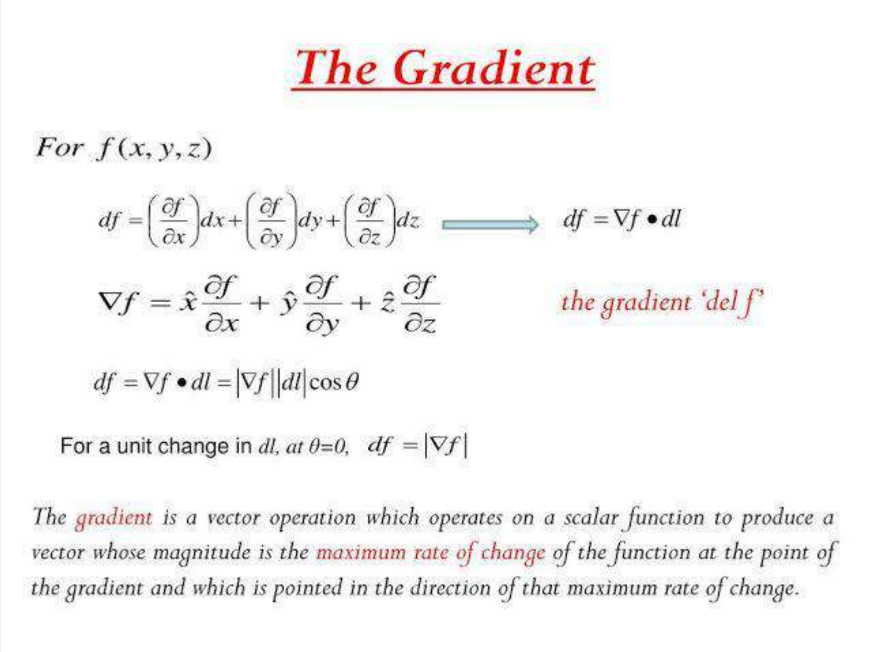
$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

This vector operator may be applied as a directive differentiator to a scalar or to a vector function of space. As the symbol $\vec{\nabla}$ denotes an invariant vector differentiator which requires an operand whether scalar or vector on which it can operate, its expression in vectorial form should only be considered as symbolic or formal. In vector analysis there are three fundamental operations with $\vec{\nabla}$ which are of physical interest. If ϕ is a scalar function and $\vec{V}$ a vector function of space these operations are as follows: (i) $\vec{\nabla}\phi$, (ii) $\vec{\nabla}\cdot\vec{V}$ and (iii) $\vec{\nabla}\times\vec{V}$.

Gradient of a scalar

Del operator is VECTOR . When it operates on a scalar function , the result will be VECTOR





The Gradient

For $f(x, y, z)$

$$df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z}\right)dz \Longrightarrow df = \nabla f \bullet dl$$

$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$ the gradient 'del f'

$$df = \nabla f \bullet dl = |\nabla f||dl|\cos\theta$$

For a unit change in dl, at $\theta = 0$, $df = |\nabla f|$

The gradient is a vector operation which operates on a scalar function to produce a vector whose magnitude is the maximum rate of change of the function at the point of the gradient and which is pointed in the direction of that maximum rate of change.

2.15 The Gradient of a Scalar Field

Certain physical quantities such as density, temperature, electric potential, etc. or any such non-directed quantity can be represented from point to point in space by a scalar point-function ϕ of the co-ordinates.

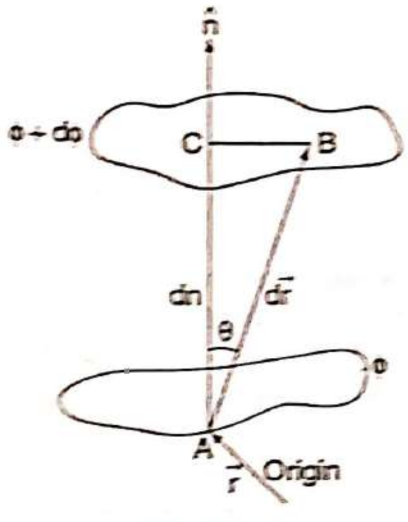


Fig. 2.15.1

The entire scalar field can be mapped out by level surfaces, the scalar function on each of which remains constant. We consider two such surfaces very close to each other. Let the two surfaces be characterised by the constant values ϕ and $\phi + d\phi$ of the scalar function. We shall examine a small portion of the two surfaces near a given point A on the surface denoted by the function ϕ [Fig. 2.15.1]. Let $\bar{r}$ and $\bar{r}+d\bar{r}$ be the radius vectors from the origin to the points A and B respectively. From the figure it follows that AC is the least distance between the two surfaces in the direction of normal unit vector $\hat{n}$ at A which is of length dn. If dr be the length of AB, the magnitude of the rate of increase of ϕ at A in the direction of AB is $\frac{\partial \phi}{\partial r}$, when the distance between the two surfaces is vanishingly small. In the direction of the unit normal vector $\hat{n}$ this rate of increase is maximum, i.e., $\frac{\partial \phi}{\partial n}$ is maximum.

It is to be noted that $\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial n}\cos\theta$. Hence, the vector $\hat{n}\frac{\partial \phi}{\partial n}$ gives the maximum rate of increase of ϕ at any point on the level surface, $\hat{n}$ being the unit vector normal to the level surface. This vector is called the gradient of ϕ at the point and is written as

$$\text{grad}\,\phi = \hat{n}\frac{\partial \phi}{\partial n}. \qquad (2.15.1)$$

Thus we see that the gradient of a scalar field is a vector field, the magnitude of which at any point is equal to the maximum rate of increase of ϕ and the direction is perpendicular to the level surface at that point.

A simple illustrative example in this case is an equipotential surface. The electric intensity X at any point is in the direction of the maximum rate of decrease of potential ϕ, i.e., normal to the equipotential surface, and has a magnitude equal to the rate of decrease of potential. Hence, the electric intensity is given by $X = -\,\text{grad}\,\phi$.

We consider the vector $\vec{\nabla}\phi$ given by

$$\vec{\nabla}\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\hat{\imath}\frac{\partial\phi}{\partial x}$, $\hat{\jmath}\frac{\partial\phi}{\partial y}$ and $\hat{k}\frac{\partial\phi}{\partial z}$ are the vector rates of increase of ϕ in the directions of the axes of X, Y and Z. We will now show that the sum of the above vector rates is equal to the gradient of ϕ. Taking scalar product on the two sides of equation (2.15.1) with an element of radius vector $d\vec{r}$, we have

$$(\text{grad}\,\phi)\cdot d\vec{r} = \frac{\partial\phi}{\partial n}\hat{n}\cdot d\vec{r} = \frac{\partial\phi}{\partial n}dr\cos\theta = \frac{\partial\phi}{\partial n}dn = d\phi$$

$\frac{\partial\phi}{\partial n}$ is the total normal rate of change of ϕ.

Now,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz.$$

$$\therefore\ (\text{grad}\,\phi)\cdot d\vec{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

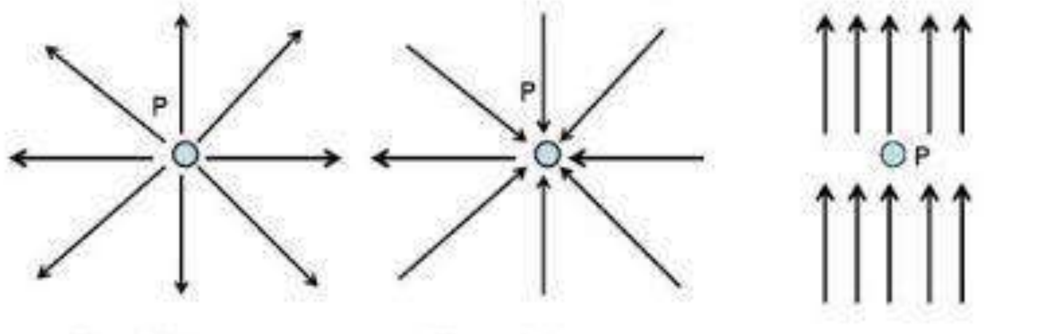
$$= \left(\vec{\nabla}\phi\right)\cdot d\vec{r}$$

$$\therefore\ \vec{\nabla}\phi = \text{grad}\,\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} \quad (2$$

Divergence of a Vector

Divergence of a vector field

- Illustration of the divergence of a vector field at point P:



Positive Divergence | Negative Divergence | Zero Divergence

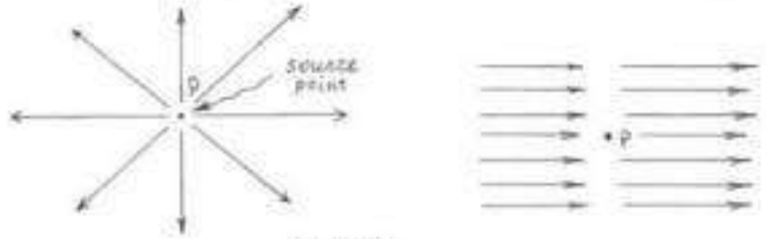
2-7 Divergence of a Vector Field

$$div\, A = \nabla\cdot\vec{A} = \lim_{\Delta v\to 0}\frac{\oint_S \vec{A}\cdot d\vec{S}}{\Delta v};$$ the divergence of $\vec{A}$ at a given point P (2-98)

where ∇v is the volume enclosed by the closed surface S in which P is located.

$\vec{A}$

Physical meaning: we may regard the divergence of the vector field at a given point as a measure of how much the field diverges or emanates from that point.



We consider a vector $\vec{A}$ at the midpoint of an infinitesimal element of volume with sides dx, dy and dz parallel to the axes OX, OY and OZ. The vector $\vec{A}$ has components of magnitudes A_x, A_y and A_z parallel to OX, OY and OZ respectively. The value of x-component of the vector at the midpoint of the left-hand face of the volume element is $A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx$. When the volume element is infinitesimally small this component of vector may be considered all over the face. Similarly, on the right-hand face the x-component of the vector is $A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx$.

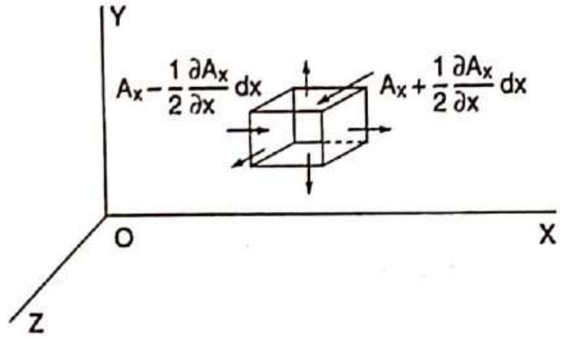


Fig. 2.17.1

Then the excess of flux leaving the element over that entering it parallel to OX is

$$\left(A_x + \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right) dy\, dz - \left(A_x - \frac{1}{2}\frac{\partial A_x}{\partial x}dx\right) dy\, dz = \frac{\partial A_x}{\partial x}dx\, dy\, dz.$$

By similar reasons the contributions to the flux parallel to OY and OZ are

$$\frac{\partial A_y}{\partial y}dx\, dy\, dz \text{ and } \frac{\partial A_z}{\partial z}dx\, dy\, dz.$$

Hence the net flux diverging from the element is

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dx\, dy\, dz.$$

The amount of this flux per unit volume is defined as the divergence of the vector $\vec{A}$ which is written as

$$\text{div } \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{2.17.1}$$

Now,

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z\right) = \vec{\nabla}\cdot\vec{A}$$

$$\therefore \text{ div } \vec{A} = \vec{\nabla}\cdot\vec{A}. \tag{2.17.2}$$

When the flux entering the volume element is equal to that leaving it, we have,

$$\text{div}\vec{A} = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{A} = 0.$$

In case of fluid motion this means that there is no accumulation of fluid anywhere. This is called the continuity equation for an incompressible fluid. As the fluid is neither created nor destroyed at any point it is said to have no sources or sinks. The lines of flow of the vector $\vec{A}$, in this case, either form closed curves or terminate upon bounding surfaces or extend to infinity. A vector satisfying the above conditions is called *solenoidal*.

Curl of a Vector

CURL OF A VECTOR (Cont'd)

For Cartesian coordinate:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x - \left[\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right]\mathbf{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z$$

The Curl of a Vector Field

The curl of $\boldsymbol{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ is

$$curl\ \vec{\boldsymbol{F}}(x,y,z) = \nabla \times \vec{\boldsymbol{F}}(x,y,z)$$

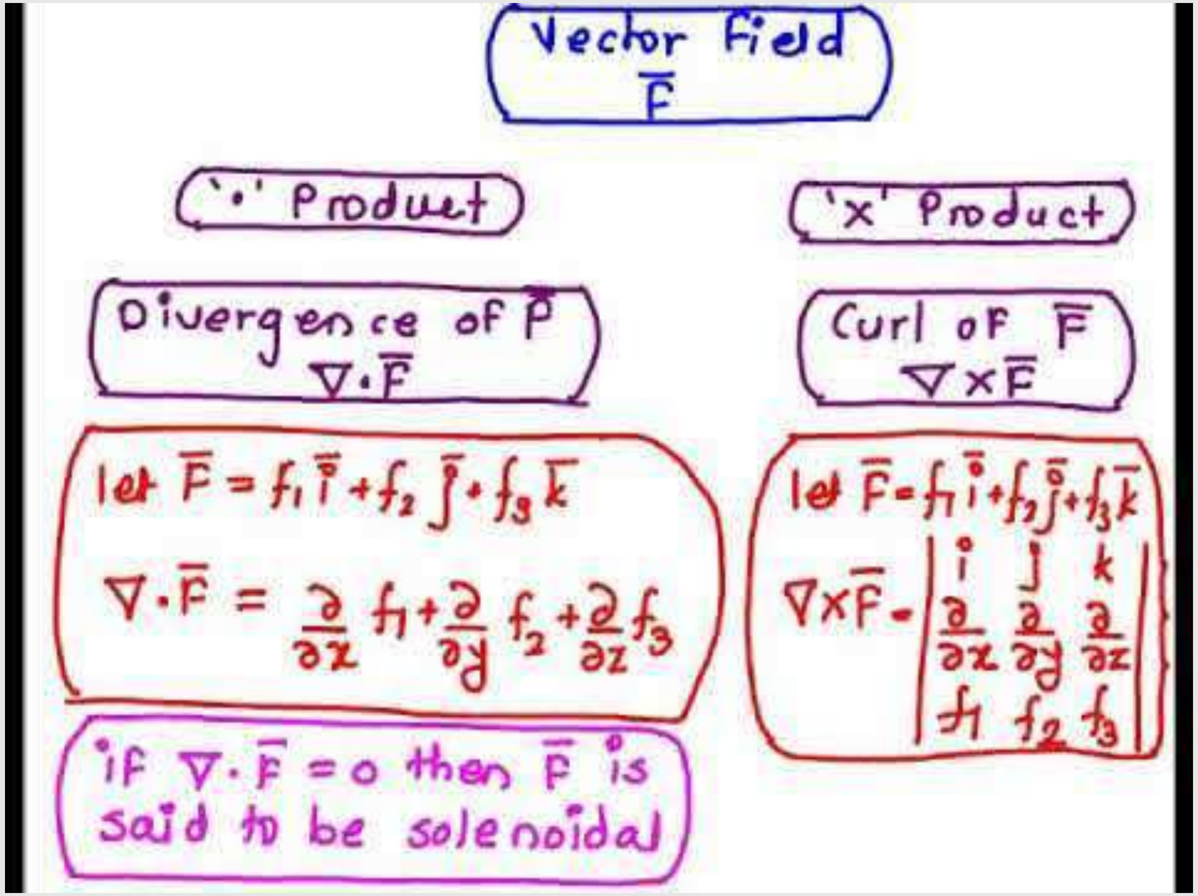
$$curl\ \vec{\boldsymbol{F}}(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$curl\ \vec{\boldsymbol{F}}(x,y,z) = \left\langle \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right), -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \right\rangle$$

∇ is a differential operator. $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

The curl of a vector field measures the rate of rotation

Solenoidal field
and
Irrotational Field



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Solved problems

3. Find $\nabla\phi$ if (a) $\phi = \ln|\mathbf{r}|$, (b) $\phi = \frac{1}{r}$.

(a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and $\phi = \ln|\mathbf{r}| = \frac{1}{2}\ln(x^2+y^2+z^2)$.

$$\nabla\phi = \tfrac{1}{2}\nabla\ln(x^2+y^2+z^2)$$

$$= \tfrac{1}{2}\left\{\mathbf{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2) + \mathbf{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2) + \mathbf{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2)\right\}$$

$$= \tfrac{1}{2}\left\{\mathbf{i}\frac{2x}{x^2+y^2+z^2} + \mathbf{j}\frac{2y}{x^2+y^2+z^2} + \mathbf{k}\frac{2z}{x^2+y^2+z^2}\right\} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2} = \frac{\mathbf{r}}{r^2}$$

(b) $$\nabla\phi = \nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \nabla\left\{(x^2+y^2+z^2)^{-1/2}\right\}$$

$$= \mathbf{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \mathbf{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \mathbf{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= \mathbf{i}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2x\right\} + \mathbf{j}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2y\right\} + \mathbf{k}\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}2z\right\}$$

$$= \frac{-x\mathbf{i}-y\mathbf{j}-z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mathbf{r}}{r^3}$$

5. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$ where c is a constant.

Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector to any point $P(x,y,z)$ on the surface. Then $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ lies in the tangent plane to the surface at P.

But $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$ or $(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = 0$

i.e. $\nabla\phi \cdot d\mathbf{r} = 0$ so that $\nabla\phi$ is perpendicular to $d\mathbf{r}$ and therefore to the surface.

7. Find an equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1,-1,2)$.

$$\nabla(2xz^2 - 3xy - 4x) = (2z^2 - 3y - 4)\,\mathbf{i} - 3x\,\mathbf{j} + 4xz\,\mathbf{k}$$

Then a normal to the surface at the point $(1,-1,2)$ is $7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$.

The equation of a plane passing through a point whose position vector is $\mathbf{r}_0$ and which is perpendicular to the normal $\mathbf{N}$ is $(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{N} = 0$. (See Chap.2, Prob.18.) Then the required equation is

$$[(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})]\cdot(7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0$$

or

$$7(x-1) - 3(y+1) + 8(z-2) = 0.$$

10. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

$$\nabla\phi = \nabla(x^2yz + 4xz^2) = (2xyz + 4z^2)\mathbf{i} + x^2z\,\mathbf{j} + (x^2y + 8xz)\mathbf{k}$$
$$= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \quad \text{at} \quad (1,-2,-1).$$

The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is

$$\mathbf{a} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Then the required directional derivative is

$$\nabla\phi \cdot \mathbf{a} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k})\cdot(\tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

Since this is positive, ϕ is increasing in this direction.

12. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2,-1,2)$.

The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

A normal to $x^2 + y^2 + z^2 = 9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

A normal to $z = x^2 + y^2 - 3$ or $x^2 + y^2 - z = 3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2 + y^2 - z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2| \cos\theta$, where θ is the required angle. Then

$$(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\cdot(4\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = |4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}|\,|4\mathbf{i} - 2\mathbf{j} - \mathbf{k}| \cos\theta$$
$$16 + 4 - 4 = \sqrt{(4)^2 + (-2)^2 + (4)^2}\,\sqrt{(4)^2 + (-2)^2 + (-1)^2} \cos\theta$$

and $\cos\theta = \dfrac{16}{6\sqrt{21}} = \dfrac{8\sqrt{21}}{63} = 0.5819$; thus the acute angle is $\theta = \arccos 0.5819 = 54^\circ 25'$.

27. Prove: (*a*) $\nabla\times(\nabla\phi) = \mathbf{0}$ (curl grad $\phi = \mathbf{0}$), (*b*) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$ (div curl $\mathbf{A} = 0$).

$$(a)\ \nabla\times(\nabla\phi) = \nabla\times\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\mathbf{k}$$

$$= \left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right)\mathbf{k} = \mathbf{0}$$

provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

30. If $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, prove $\boldsymbol{\omega} = \frac{1}{2}\,\text{curl}\,\mathbf{v}$ where $\boldsymbol{\omega}$ is a constant vector.

$$\text{curl}\,\mathbf{v} = \nabla\times\mathbf{v} = \nabla\times(\boldsymbol{\omega}\times\mathbf{r}) = \nabla\times\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla\times\left[(\omega_2 z - \omega_3 y)\mathbf{i} + (\omega_3 x - \omega_1 z)\mathbf{j} + (\omega_1 y - \omega_2 x)\mathbf{k}\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1\text{[illegible]} & \text{[illegible]}x \end{vmatrix} = 2(\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}) = 2\boldsymbol{\omega}.$$

31. If $\nabla\cdot\mathbf{E}=0$, $\nabla\cdot\mathbf{H}=0$, $\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{H}}{\partial t}$, $\nabla\times\mathbf{H}=\dfrac{\partial\mathbf{E}}{\partial t}$, show that $\mathbf{E}$ and $\mathbf{H}$ satisfy $\nabla^2 u=\dfrac{\partial^2 u}{\partial t^2}$.

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla\times\left(-\frac{\partial\mathbf{H}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{E}}{\partial t}\right) = -\frac{\partial^2\mathbf{E}}{\partial t^2}$$

By Problem 29, $\nabla\times(\nabla\times\mathbf{E}) = -\nabla^2\mathbf{E}+\nabla(\nabla\cdot\mathbf{E}) = -\nabla^2\mathbf{E}$. Then $\nabla^2\mathbf{E}=\dfrac{\partial^2\mathbf{E}}{\partial t^2}$.

Similarly, $\nabla\times(\nabla\times\mathbf{H}) = \nabla\times\left(\dfrac{\partial\mathbf{E}}{\partial t}\right) = \dfrac{\partial}{\partial t}(\nabla\times\mathbf{E}) = \dfrac{\partial}{\partial t}\left(-\dfrac{\partial\mathbf{H}}{\partial t}\right) = -\dfrac{\partial^2\mathbf{H}}{\partial t^2}$.

But $\nabla\times(\nabla\times\mathbf{H}) = -\nabla^2\mathbf{H}+\nabla(\nabla\cdot\mathbf{H}) = -\nabla^2\mathbf{H}$. Then $\nabla^2\mathbf{H}=\dfrac{\partial^2\mathbf{H}}{\partial t^2}$.

The given equations are related to *Maxwell's equations of electromagnetic theory*. The equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{\partial^2 u}{\partial t^2}$ is called the *wave equation.*

Assignments

42. If $\phi = 2xz^4 - x^2y$, find $\nabla\phi$ and $|\nabla\phi|$ at the point $(2,-2,-1)$. *Ans.* $10\mathbf{i} - 4\mathbf{j} - 16\mathbf{k}$, $2\sqrt{93}$

43. If $\mathbf{A} = 2x^2\,\mathbf{i} - 3yz\,\mathbf{j} + xz^2\,\mathbf{k}$ and $\phi = 2z - x^3y$, find $\mathbf{A}\cdot\nabla\phi$ and $\mathbf{A}\times\nabla\phi$ at the point $(1,-1,1)$.
Ans. 5, $7\mathbf{i} - \mathbf{j} - 11\mathbf{k}$

44. If $F = x^2z + e^{y/x}$ and $G = 2z^2y - xy^2$, find (a) $\nabla(F+G)$ and (b) $\nabla(FG)$ at the point $(1,0,-2)$.
Ans. (a) $-4\mathbf{i} + 9\mathbf{j} + \mathbf{k}$, (b) $-8\mathbf{j}$

45. Find $\nabla|\mathbf{r}|^3$. *Ans.* $3r\,\mathbf{r}$

71. Evaluate $\operatorname{div}(2x^2z\,\mathbf{i} - xy^2z\,\mathbf{j} + 3yz^2\,\mathbf{k})$. *Ans.* $4xz - 2xyz + 6yz$

72. If $\phi = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$, find $\nabla^2\phi$. *Ans.* $6z + 24xy - 2z^3 - 6y^2z$

73. Evaluate $\nabla^2(\ln r)$. *Ans.* $1/r^2$

74. Prove $\nabla^2 r^n = n(n+1)r^{n-2}$ where n is a constant.

75. If $\mathbf{F} = (3x^2y - z)\mathbf{i} + (xz^3 + y^4)\mathbf{j} - 2x^3z^2\,\mathbf{k}$, find $\nabla(\nabla\cdot\mathbf{F})$ at the point $(2,-1,0)$. *Ans.* $-6\mathbf{i} + 24\mathbf{j} - 32\mathbf{k}$

76. If $\boldsymbol{\omega}$ is a constant vector and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$, prove that $\operatorname{div}\mathbf{v} = 0$.

77. Prove $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\cdot\nabla\psi + \psi\nabla^2\phi$.

78. If $U = 3x^2y$, $V = xz^2 - 2y$ evaluate $\operatorname{grad}[(\operatorname{grad} U)\cdot(\operatorname{grad} V)]$. *Ans.* $(6yz^2 - 12x)\mathbf{i} + 6xz^2\,\mathbf{j} + 12xyz\,\mathbf{k}$

79. Evaluate $\nabla\cdot(r^3\,\mathbf{r})$. *Ans.* $6r^3$

80. Evaluate $\nabla\cdot[r\,\nabla(1/r^3)]$. *Ans.* $3r^{-4}$

81. Evaluate $\nabla^2[\nabla\cdot(\mathbf{r}/r^2)]$. *Ans.* $2r^{-4}$

82. If $\mathbf{A} = \mathbf{r}/r$, find $\operatorname{grad}\operatorname{div}\mathbf{A}$. *Ans.* $-2r^{-3}\,\mathbf{r}$

96. If $\mathbf{A} = yz^2\,\mathbf{i} - 3xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}$, $\mathbf{B} = 3x\,\mathbf{i} + 4z\,\mathbf{j} - xy\,\mathbf{k}$ and $\phi = xyz$, find
(*a*) $\mathbf{A}\times(\nabla\phi)$, (*b*) $(\mathbf{A}\times\nabla)\phi$, (*c*) $(\nabla\times\mathbf{A})\times\mathbf{B}$, (*d*) $\mathbf{B}\cdot\nabla\times\mathbf{A}$.

Ans. (*a*) $-5x^2yz^2\,\mathbf{i} + xy^2z^2\,\mathbf{j} + 4xyz^3\,\mathbf{k}$
(*b*) $-5x^2yz^2\,\mathbf{i} + xy^2z^2\,\mathbf{j} + 4xyz^3\,\mathbf{k}$ (same as (*a*))
(*c*) $16z^3\,\mathbf{i} + (8x^2yz - 12xz^2)\mathbf{j} + 32xz^2\,\mathbf{k}$ (*d*) $24x^2z + 4xyz^2$

97. Find $\mathbf{A}\times(\nabla\times\mathbf{B})$ and $(\mathbf{A}\times\nabla)\times\mathbf{B}$ at the point $(1,-1,2)$, if $\mathbf{A} = xz^2\,\mathbf{i} + 2y\,\mathbf{j} - 3xz\,\mathbf{k}$ and $\mathbf{B} = 3xz\,\mathbf{i} + 2yz\,\mathbf{j} - z^2\,\mathbf{k}$.

Ans. $\mathbf{A}\times(\nabla\times\mathbf{B}) = 18\mathbf{i} - 12\mathbf{j} + 16\mathbf{k}$, $(\mathbf{A}\times\nabla)\times\mathbf{B} = 4\mathbf{j} + 76\mathbf{k}$

98. Prove $(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v})$.

99. Prove $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$.

100. Prove $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B})$.

101. Prove $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$.

84. Prove that the vector $\mathbf{A} = 3y^4z^2\,\mathbf{i} + 4x^3z^2\,\mathbf{j} - 3x^2y^2\,\mathbf{k}$ is solenoidal.

85. Show that $\mathbf{A} = (2x^2 + 8xy^2z)\mathbf{i} + (3x^3y - 3xy)\mathbf{j} - (4y^2z^2 + 2x^3z)\mathbf{k}$ is not solenoidal but $\mathbf{B} = xyz^2\,\mathbf{A}$ is solenoidal.

* **19.** Prove that $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$. If $\vec{A}$ and $\vec{B}$ are irrotational, show that $\vec{A}\times\vec{B}$ is solenoidal.

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